Lineability and spaceability on vector-measure spaces

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Abstract

It is proved that if X is infinite dimensional, then there exists an infinite dimensional space of X-valued measures which have infinite variation on sets of positive Lebesgue measure. In term of spaceability, it is also shown that $ca(\mathcal{B}, \lambda, X) \setminus M_\sigma$, measures with not σ-finite variation, contains a closed subspace. Other considerations are made for the space of vector measures whose range are neither closed nor convex. All of those results extend in some sense theorems in [7].

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1 Brief introduction and results

We begin by recalling the following relatively new concepts related to the "algebraical size" of subsets of Banach spaces.

Definition 1.1 (Gurariy, 1991) *A subset M of a Banach space is said to be*

1. n-lineable *if $M \cup \{0\}$ contains an n-dimensional vector subspace;*
2. lineable *if $M \cup \{0\}$ contains an infinite dimensional vector subspace;*
3. dense-lineable *if $M \cup \{0\}$ contains an infinite dimensional dense vector subspace;*

4. spaceable *if $M \cup \{0\}$ contains an infinite dimensional closed vector subspace.*

Let $I = [0,1]$ be the unit interval and let $\mathcal{B}$ denote the σ-algebra of all Borel subsets of I. Also, let λ be the Lebesgue measure on I. For a Banach space X we let $ca(\mathcal{B}, \lambda, X)$ stand for the space of all vector measures $\mu : \mathcal{B} \longrightarrow X$ which are countable additive and absolutely continuous with respect to λ. Then $ca(\mathcal{B}, \lambda, X)$ is a Banach space endowed with the norm

$$\|\mu\|_{ca} = \sup_{A \in \mathcal{B}} \|\mu(A)\|_X.$$

Let us also recall that the *variation* of a vector measure is defined by

$$|\mu|(A) = \sup\{\sum_{i=1}^{n} \|\mu(A_i)\| \ : \ A_i \text{ pairwise disjoint } \bigcup_{i=1}^{n} A_i = A\}.$$

In the sequel, we shall use the following notation for the space

$$cabv(\mathcal{B}, \lambda, X) = \{\mu \in ca(\mathcal{B}, \lambda, X) \ : \ |\mu| \text{ is finite}\}.$$

Then $cabv(\mathcal{B}, \lambda, X)$ endowed with the variation norm $|\cdot|$ (i.e., the norm is $|\mu|(I)$) is a Banach space.

In [7] it is proved the following.

Theorem 1.2 *Let $(\mathcal{B}, \lambda)$ be the Lebesgue measure space on the unit interval, let $1 \leq p < \infty$. Then the set of ℓ_p-valued measures with relatively compact range such that their variation measures take the value infinity on every non-null set is lineable in $ca(\mathcal{B}, \lambda, \ell_p)$.*

We would like to note that the above result holds for any infinite dimensional Banach space.

Following [3], let us denote by M_σ the subspace of $ca(\mathcal{B}, \lambda, X)$ of all measures μ such that $|\mu|$ is σ-finite. Let ρ be the metrizable vector topology on M_σ defined by the base $\{V_n \ \ n \in \mathbb{N}\}$, where

$$V_n = \left\{ \mu \in ca(\mathcal{B}, \lambda, X) : \begin{array}{c} \|\mu\| \leq \frac{1}{2^n} \text{ and there exists } E \in \mathcal{B} \text{ with} \\ \lambda(E) \leq \frac{1}{2^n} \text{ and } |\mu|(I \setminus E) \leq \frac{1}{2^n} \end{array} \right\}$$

It is easy to see that (M_σ, ρ) is complete.

At this stage, we are ready to show the following.

Theorem 1.3 *Let X be any infinite dimensional Banach space and let $(\mathcal{B}, \lambda)$ be the Lebesgue measure space on the unit interval. Then the set of X-valued measures with relatively compact range such that their variation measures take the value infinity on every non-null set is lineable in $ca(\mathcal{B}, \lambda, X)$.*

Proof. Let $(A_n)_n$ be a sequence of Borel sets in I such that

- $I = \bigcup_n A_n$;
- $A_n \cap A_m = \emptyset$ for $n \neq m$;
- $\lambda(A_n) > 0$ for every $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let P_n be the subspace of $ca(\mathcal{B}, \lambda, X)$ of all simple measures μ of kind

$$\mu(A) = \sum_{finite} \lambda(A \cap A_n) x_k, \quad A \in \mathcal{B}.$$

By a Dvoretzy-Rogers trick, it is not hard to show that

$$\overline{P_n}^{ca(\mathcal{B},\lambda,X)} \not\subseteq M_\sigma \quad \text{(see [3])}.$$

Therefore, if we pick $\mu_n \in \overline{P_n}^{ca(\mathcal{B},\lambda,X)} \setminus M_\sigma$, we have that

- all the μ_n's have relatively compact range and their variation measures take the value infinity on every non-null set (see [3, Theorem 2]),
- all linear combinations of the μ_n's have relatively compact range and their variation measures take the value infinity on every non-null set, and
- the μ_n's are linearly independent (because they have disjoint supports).

■

Now, we would like to deal with the following question.

Question 1.4 *Is $ca(\mathcal{B}, \lambda, X) \setminus M_\sigma$ spaceable?*

It is proved in [6] (see also [1]) the following remarkable result.

Theorem 1.5 *Let Z_n ($n \in \mathbb{N}$) be Banach spaces and X a Fréchet space. Let $T_n : Z_n \longrightarrow X$ be continuous linear operators and Y the linear span of $\bigcup_n T_n(Z_n)$. If Y is not closed in X, then the complement $X \setminus Y$ is spaceable.*

Before going on, let us recall some standard concepts. For a sequence of Banach spaces $(X_n, \|\cdot\|_n)$ such that all X_n's are (isomorphic to) a closed subspace of a bigger Banach space $\mathcal{X}$, consider

$$\left(\bigoplus_{n\in\mathbb{N}} X_n\right)_c = \{x_n \in X_n : \lim_{n\to\infty} x_n \text{ exists in } \mathcal{X}\},$$

endowed with the norm

$$\|(x_n)_n\| = \sup_n \|x_n\|_n.$$

Then $\left(\bigoplus_{n\in\mathbb{N}} X_n\right)_c$ is a Banach space.

We are ready to state the main Theorem of this note.

Theorem 1.6 *If X is infinite dimensional, then $ca(\mathcal{B}, \lambda, X) \setminus M_\sigma$ is spaceable.*

Proof. Let us fix a sequence $(A_n)_n \subseteq \mathcal{B}$ such that

1. $A_n \subseteq A_{n+1}$, for each $n \in \mathbb{N}$,
2. $\lambda(A_{n+1} \setminus A_n) > 0$, for each $n \in \mathbb{N}$,

3. $\bigcup_{n\in\mathbb{N}} A_n = I$.

Let $\Sigma_n = \{E \cap A_n : \; E \in \mathcal{B}\}$ be the σ-algebra generated by A_n. Since, for each $n \in \mathbb{N}$, we can see $(cabv(\Sigma_n, \lambda, X), \|\cdot\|_{ca})$ as a closed subspace of $(cabv(\mathcal{B}, \lambda, X), \|\cdot\|_{ca})$ (via the natural map that associates to each $\mu \in (cabv(\Sigma_n, \lambda, X), \|\cdot\|_{ca})$ the measure that is equal to μ on Σ_n and zero outside A_n), then we can consider the Banach space

$$\left(\bigoplus_{n\in\mathbb{N}} (cabv(\Sigma_n, \lambda, X, \; \|\cdot\|_{ca})\right)_c.$$

Let us define

$$\mathcal{M} = \left\{ (\mu_n)_n \in \left(\bigoplus_{n\in\mathbb{N}} (cabv(\Sigma_n, \lambda, X), \|\cdot\|_{ca})\right)_c \; : \; \mu_{n+1}|_{\Sigma_n} = \mu_n \right\}.$$

Let us show that $\mathcal{M}$ is a closed subspace of $(\bigoplus_{n\in\mathbb{N}} (cabv(\Sigma_n, \lambda, X), \|\cdot\|_{ca}))_c$.

Let $(\overline{\mu}^p)_p \subseteq \mathcal{M}$ (where $\overline{\mu}^p = (\mu_n^p)_n$ for each $p \in \mathbb{N}$) be a sequence such that

$$\lim_{p\to\infty} = \overline{\mu} = (\mu_n)_n \in \left(\bigoplus_{n\in\mathbb{N}} (cabv(\Sigma_n, \lambda, X), \|\cdot\|_{ca})\right)_c;$$

explicitly,

$$\sup_n \sup_A \|\mu_n^p(A) - \mu_n(A)\| \stackrel{p\to\infty}{\longrightarrow} 0.$$

Let $A \in \Sigma_n$. Since $\mu_{n+1}^p(A) = \mu_n^p(A)$ we have

$$\|\mu_{n+1}(A) - \mu_n(A)\| \leq \|\mu_{n+1}^p(A) - \mu_{n+1}(A)\| + \|\mu_n^p(A) - \mu_n(A)\| \stackrel{p\to\infty}{\longrightarrow} 0.$$

Namely, $\overline{\mu} \in \mathcal{M}$. Therefore, $\mathcal{M}$ is a Banach space.

Let us define

$$T : \mathcal{M} \longrightarrow (ca(\mathcal{B}, \lambda, X), \; \|\cdot\|_{ca})$$

defined by

$$T((\mu_n)_n)(A) = \lim_{n\to\infty} \mu_n(A \cap A_n) \quad \forall A \in \mathcal{B}.$$

Let us prove that T is a continuous linear operator such that $T(\mathcal{M}) = M_\sigma$.

First, let us note that T is well defined. Indeed, let $(E_k)_k \subseteq \mathcal{B}$ be a disjoint sequence of sets. Then

$$\begin{aligned} T((\mu_n)_n)(\bigcup_k E_k) &= \lim_{n\to\infty} \mu_n((\bigcup_k E_k) \cap A_n) \\ &= \lim_{n\to\infty} \mu_n(\bigcup_k (E_k \cap A_n)) \\ &= \lim_{n\to\infty} \sum_k \mu_n(E_k \cap A_n) \\ &= \sum_k \lim_{n\to\infty} \mu_n(E_k \cap A_n) \\ &= \sum_k T((\mu_n)_n)(E_k), \end{aligned}$$

since we have convergence with respect to the semivariation norm $\|\cdot\|_{ca}$. Moreover, it is evident that $T((\mu_n)_n)$ is λ-continuous.

The linearity follows directly from the definition.

For the continuity,

$$\begin{aligned}
\|T((\mu_n)_n)\|_{ca} &= \sup_{A\in\mathcal{B}} \|T((\mu_n)_n)(A)\| \\
&= \sup_{A\in\mathcal{B}} \| \lim_{n\to\infty} \mu_n(A\cap A_n)\| \\
&\leq \sup_{A\in\mathcal{B}} \lim_{n\to\infty} \|\mu_n(A\cap A_n)\| \\
&= \sup_{n\in\mathbb{N}} \sup_{A\in\mathcal{B}} \|\mu_n(A\cap A_n)\| \\
&= \|(\mu_n)_n\|_{\mathcal{M}}.
\end{aligned}$$

From the equality $T(\mathcal{M}) = M_\sigma$, let us first note that $T((\mu_n)_n)$ is a measure of σ-finite variation. Indeed, by construction, for each $s \in \mathbb{N}$

$$\begin{aligned}
|T((\mu_n)_n)|(A_s) &\leq \lim_{n\to\infty} |\mu_n|(A_s) \\
\text{(by the definition on } \mathcal{M}) \quad &= |\mu_s|(A_s) \\
&< +\infty.
\end{aligned}$$

Moreover, if $\mu \in M_\sigma$, since $|\mu|$ is σ-finite, consider an increasing sequence $(C_n)_n$ such that

$$\bigcup_n C_n = I \text{ and } |\mu|(C_n) < +\infty, \text{ for all } n \in \mathbb{N};$$

now, take $\mu_n \in cabv(\Sigma_n, \lambda, x)$ defined by $\mu_n(A\cap A_n) = \mu(A\cap A_n\cap C_n)$. Then, by construction, $(\mu_n)_n \in \mathcal{M}$ and we have

$$T((\mu_n)_n) = \mu.$$

It was already observed in [3] that M_σ, respect to the complete metric ρ, is not closed in $(ca(\mathcal{B},\lambda,X), \|\cdot\|_{ca})$. Since the topology generated by ρ is stronger than the norm topology $\|\cdot\|_{ca}$, we have that $(M_\sigma, \|\cdot\|_{ca})$ is not closed in $(ca(\mathcal{B},\lambda,X), \|\cdot\|_{ca})$ either. The proof is concluded by simply applying Theorem 1.5 above. ∎

Let us recall the following definition (see [2]).

Definition 1.7 *Let (Ω,Σ) be a measurable space, λ a positive measure on Σ, and X an infinite dimensional Banach space. A measure $\mu \in ca(\lambda, X)$ is said to be* injective *when for each $\phi,\psi \in L_\infty(\lambda)$ the following condition holds:*

$$\textit{if} \quad \int \phi d\mu = \int \psi d\mu \quad \textit{then} \quad \phi = \psi \quad \lambda - a.e.$$

In [7], using a nice construction, the authors were able to show the following

Theorem 1.8 *Let λ be the Lebesgue measure on the Borel sets in $[0,1]$ and X an infinite dimensional Banach space. Then the set of injective measures is lineable in $ca(\lambda, X)$.*

At this point we are interested in the spaceability of the set of injective measures. In [8], A. Wilansky proved the following general criterion to have spaceability

Theorem 1.9 *Let E be a Banach space. If F is a closed infinite codimensional vector subspace of a Banach space E, then $E \setminus F$ is spaceable.*

We would like to use this criterion to note the following.

Theorem 1.10 *Let λ the Lebesgue measure on the Borel sets in $[0,1]$, and X be an infinite dimensional Banach space. Then the set of injective measures is spaceable in $ca(\lambda, X)$.*

Before providing the proof, we need the following lemma.

Lemma 1.11 *The space*

$$\mathcal{NI} = \{\mu \in ca(\lambda, X) : \ \mu \text{ is not injective}\}$$

is a closed subspace of $ca(\lambda, X)$.

Proof. We will provide two different proofs:

1^{th} way To show that it is closed it is enough to note the following: a measure $\mu \in ca(\lambda, X)$ is injective if and only if the integral operator associate to μ

$$T_\mu : L_\infty(\lambda) \longrightarrow X$$
$$T_\mu(f) = \int f d\mu,$$

is injective.

Suppose that $(\mu_n)_n \subseteq \mathcal{NI}$ converges to $\mu \in ca(\lambda, X)$, and μ is injective. Therefore,

$$L_\infty(\lambda)^* = \overline{T_\mu^*(X^*)}^{weak^*}$$

Since $(\mu_n)_n$ converges to μ, we have that

$$\overline{T_\mu^*(X^*)}^{weak^*} \subseteq \bigcup_{n\in\mathbb{N}} \overline{T_{\mu_n}^*(X^*)}^{weak^*}.$$

Thus, there must exists $\overline{n} \in \mathbb{N}$ such that

$$weak^* - int\left(T_{\mu_{\overline{n}}}^*(X^*)\right) \neq \emptyset.$$

Since $T_{\mu_{\overline{n}}}^*(X^*)$ is a vector subspace, that would implies

$$\overline{T_{\mu_{\overline{n}}}^*(X^*)}^{weak^*} = L_\infty^*(\lambda).$$

Against the fact that $\mu_{\overline{n}} \in \mathcal{NI}$.

2^{th} way It is well known that $\mu \in \mathcal{NI}$ if and only if for each $B \in \mathcal{B}$, $\{\mu(A \cap B) : A \in \mathcal{B}\}$ is convex and weakly compact (see [5]). However, the limit of a sequence of non-empty convex closed sets in the Hausdorff metric it is still a non-empty convex closed (see [4, 4.3.11]). Moreover, since $\mu_n \overset{n\to\infty}{\longrightarrow} \mu$ implies $\{\mu_n(A \cap B) : A \in \mathcal{B}\} \overset{n\to\infty}{\longrightarrow} \{\mu(A \cap B) : A \in \mathcal{B}\}$ in the Hausdorff metric, we obtain that if each $\{\mu_n(A \cap B) : A \in \mathcal{B}\}$ is convex, weakly compact, and

$$\mu_n \overset{n\to\infty}{\longrightarrow} \mu,$$

then $\{\mu(A \cap B) : A \in \mathcal{B}\}$ is convex and weakly compact too.

From what we said above, it follows that $\mathcal{NI}$ is a closed subspace of $ca(\lambda, X)$.

■

Proof of Theorem 1.10. From Lemma 1.11, we have that $\mathcal{NI}$ is a closed subspace of $ca(\lambda, X)$ of infinite codimension. To show that the quotient $ca(\lambda, X)/\mathcal{NI}$ is infinite dimensional, it is sufficient to use a similar construction as in the proof of [7, Theorem 2.4]. Then Theorem 1.9 applies. ■

Since it is well known that every injective measure has range neither closed nor convex, we finally obtain the following corollary.

Corollary 1.12 *Let λ be the Lebesgue measure on the Borel sets in $[0,1]$, and X an infinite dimensional Banach space. Then the set of measures whose range is neither closed nor convex is spaceable in $ca(\lambda, X)$.*

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